Fuzzy Sets for a Declarative Description of Multi-adjoint Logic Programming

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Abstract. A powerful research line in the design of declarative languages consists in the introduction of expressive resources with a fuzzy taste on their cores, in order to provide comfortable computational constructs for easily solving real-world scientific/engineering problems. Into the fuzzy logic programming arena, the so-called *multi-adjoint approach* (MALP in brief) has emerged as an interesting paradigm for which our research group has developed during the last years the $\mathcal{FLOPER}$ programming environment and the *FuzzyXPath* application in the field of the semantic web. Since the practicality of declarative languages is strongly dependent of their theoretical foundations, here we focus on topics related with the declarative semantics of the MALP framework. So, under an innovative point of view relying on fuzzy sets theory, in this paper we re-formulate in a very simple and elegant way our original model theory-based notions of *least fuzzy Herbrand model* and *(fuzzy) correct answer*. Apart for simplifying the proofs relating these concepts, our results are nicely strengthened with homologous ones in the field of pure logic programming, but largely surpassing them thanks to the fuzzy dimension of the MALP language.

Keywords: Fuzzy sets and fuzzy logic, Fuzzy logic programming, Fuzzy Herbrand model, Fuzzy correct answers, Soundness, Fuzzy information systems.

1 Introduction

There exist a lot of contributions in the specialized literature related to fuzzy logic programming which pay attention to declarative (fix-point, model-theoretic, etc.) semantics which surprisingly make not explicit use of fuzzy sets. In this paper we provide a declarative description, based on fuzzy sets, of the least Herbrand model and correct answer for MALP programs.

In what follows, we present a short summary of the main features of our language (we refer the reader to [8,9,10] for a complete formulation, including completeness and other correctness properties). We work with a first order language, $\mathcal{L}$, containing variables, function symbols, predicate symbols, constants,

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quantifiers and several (arbitrary) connectives to increase language expressiveness. In our fuzzy setting, we use *implication connectives* ($\leftarrow_1, \leftarrow_2, \ldots, \leftarrow_m$) and also other connectives: *conjunctions* (denoted by $\wedge_1, \wedge_2, \ldots, \wedge_k$), *disjunctions* ($\vee_1, \vee_2, \ldots, \vee_l$)[1] and *aggregators* (usually denoted by $@_1, @_2, \ldots, @_n$) which are used to combine/propagate truth values through the rules. The general definition of an n-ary aggregator connective $@$ (that extends conjunctions, disjunctions) states for its truth function $\dot{@}$, that $\dot{@} : L^n \rightarrow L$ is required to be monotone and fulfills $\dot{@}(\top, \ldots, \top) = \top$, $\dot{@}(\bot, \ldots, \bot) = \bot$[2]. Although the connectives $\wedge_i$, $\vee_i$ and $@_i$ are binary operators, we usually generalize them as functions with an arbitrary number of arguments.

Additionally, our language $\mathcal{L}$ contains the elements of a *multi-adjoint lattice*, $(L, \leq, \leftarrow_1, \&_1, \ldots, \leftarrow_n, \&_n)$ (see Definition 3), equipped with a collection of adjoint pairs $(\leftarrow_i, \&_i)$, where each $\&_i$ is a conjunctor intended to the evaluation of *modus ponens*.

A *rule* is a logic formula $H \leftarrow_i \mathcal{B}$, where H is an atomic formula (called the *head*) and $\mathcal{B}$ (which is called the *body*) is a formula built from atomic formulas $B_1, \ldots, B_n$ $(n \geq 0)$, truth values of L and conjunctions, disjunctions and aggregators. Rules with an empty body are called *facts*. A *goal* is a body submitted as a query to the system. Variables in a rule are assumed to be governed by universal quantifier and in a goal by existential quantifier. A *multi-adjoint formula* is a rule or a goal. A *multi-adjoint logic program* $\mathcal{P}$ is a set of pairs $\mathcal{R} : \langle R; v \rangle$, where R is a (logic) rule and v is a *truth degree* (a value of L) expressing the confidence which the user of the system has in the truth of the R.

In order to describe the procedural semantics of the multi-adjoint logic language, in the following we denote by $\mathcal{C}[A]$ a formula where A is a sub-expression (usually an atom) which occurs in the –possibly empty– context $\mathcal{C}[]$ whereas $\mathcal{C}[A/A']$ means the replacement of A by A' in context $\mathcal{C}[]$. Moreover, $\mathcal{V}ar(s)$ denotes the set of distinct variables occurring in the syntactic object s, $\theta[\mathcal{V}ar(s)]$ refers to the substitution obtained from θ by restricting its domain to $\mathcal{V}ar(s)$ and $mgu(E)$ denotes the *most general unifier* of an equation set E. In the next definition, we always consider that A is the selected atom in goal $\mathcal{Q}$ and L is the multi-adjoint lattice associated to $\mathcal{P}$.

Definition 1 (Admissible Steps). *Let $\mathcal{Q}$ be a goal and let σ be a substitution. The pair $\langle \mathcal{Q}; \sigma \rangle$ is a* state. *Given a program $\mathcal{P}$, an* admissible computation *is formalized as a state transition system, whose transition relation $\rightarrow_{AS}$ is the smallest relation satisfying the following* admissible rules*:*

1) $\langle \mathcal{Q}[A]; \sigma \rangle \rightarrow_{AS} \langle (\mathcal{Q}[A/v \&_i \mathcal{B}])\theta; \sigma\theta \rangle$ *if* $\theta = mgu(\{H = A\})$, $\langle H \leftarrow_i \mathcal{B}; v \rangle$ *in* $\mathcal{P}$ *and* $\mathcal{B}$ *is not empty.*
2) $\langle \mathcal{Q}[A]; \sigma \rangle \rightarrow_{AS} \langle (\mathcal{Q}[A/v])\theta; \sigma\theta \rangle$ *if* $\theta = mgu(\{H = A\})$, $\langle H \leftarrow_i; v \rangle$ *in* $\mathcal{P}$.
3) $\langle \mathcal{Q}[A]; \sigma \rangle \rightarrow_{AS} \langle (\mathcal{Q}[A/\bot]); \sigma \rangle$ *if there is no rule in* $\mathcal{P}$ *whose head unifies with* A *(this case copes with possible unsuccessful branches).*

[1] We assume that $\wedge_i$ is a t-norm, $\vee_i$ is a t-conorm, $\leftarrow_i$ is a implication, as conceived in [15].

[2] L is a lattice according to the later Definition 3 and $\top = sup(L), \bot = inf(L)$.

Definition 2 (Admissible Derivation). *Let $\mathcal{P}$ be a program with an associated multi-adjoint lattice $(L, \leq)$ and let $\mathcal{Q}$ be a goal. An* admissible derivation *$\langle \mathcal{Q}; id\rangle \rightarrow^{*}_{AS} \langle \mathcal{Q}'; \theta\rangle$ is an arbitrary sequence of admissible steps.*

When $\mathcal{Q}'$ is a formula not containing atoms and $r \in L$ is the result of interpreting $\mathcal{Q}'$ in $(L, \leq)$, the pairs $\langle \mathcal{Q}'; \sigma\rangle$ and $\langle r; \sigma\rangle$, where $\sigma = \theta[\mathcal{V}ar(\mathcal{Q})]$, are called admissible computed answer *(a.c.a.) and* fuzzy computed answer *(f.c.a.), respectively (see [4] for details).*

Moreover, in the MALP framework [10,8,9,7], each program has its own associated multi-adjoint lattice, that we define in the following, and each program rule is "weighted" with an element of this one.

Definition 3. *A* multi-adjoint lattice *is a tuple $(L, \leq, \leftarrow_1, \&_1, \ldots, \leftarrow_n, \&_n)$ such that:*

i) *$(L, \leq)$ is a complete lattice, i.e., for all $S \subset L$, exist* $\inf(S)$ *and* $\sup(S)$[3].
ii) *$(\leftarrow_i, \&_i)$ is an* adjoint pair *in $(L, \leq)$, namely:*
  1) *$\&_i$ is increasing in both arguments, for all i, $i \in \{1, \ldots, n\}$.*
  2) *$\leftarrow_i$ is increasing in the first argument and decreasing in the second, for all i.*
  3) *$x \leq (y \leftarrow_i z)$ iff $(x \&_i z) \leq y$, for any $x, y, z \in L$ (adjoint property).*
iii) *$\top \&_i v = v \&_i \top = v$, for all $v \in L, i \in \{1, \ldots, n\}$, where $\top = sup(L)$.*

We refer the reader to [13] where we focus on two relevant mathematical concepts for this kind of domains useful for evaluating multi-adjoint logic programs, and, on the one side, we adapt the classical notion of *Dedekind-MacNeille completion* in order to relax some usual hypothesis on such kind of ordered sets.

The structure of this paper is as follows. The notion of least fuzzy Herbrand model by using fuzzy sets is presented in Section 2. Next, in Section 3 we focus on fuzzy correct answers expressed again in terms of fuzzy sets and moreover, we then prove the soundness property of the framework. Section 4 summarizes some preliminary results of logical consequences after being reformulated by means of fuzzy sets. Finally, Section 5 concludes with our on-going work.

2 Fuzzy Sets and Least Fuzzy Herbrand Model

The concept of fuzzy set, due to [21], frequently occurs when we tend to organize, summarize and generalize knowledge about objects [16]. On this concept is based the theory of uncertainty with classic references on fuzzy logic programming [14,17,18,19,20].

In this section, we use the theory of fuzzy sets in order to define, for the first time in literature, the notion of least fuzzy Herbrand model as a certain fuzzy subset of the Herbrand base. We start the development of contents with two basic notions, namely, the concept of fuzzy set and the one of L-fuzzy set.

[3] Then, it is a bounded lattice, that is, it has bottom and top elements, denoted by $\bot$ and $\top$, respectively.

Definition 4. *[16] A* fuzzy set *A of a (crisp or ordinary) set $\mathcal{U}$, may be represented as a set of ordered pairs with first component $x \in \mathcal{U}$ and second component its* degree of membership $\mu_A(x)$[4]*, that is, $A = \{x|\mu_A(x) : \mu_A(x) \neq 0, x \in \mathcal{U}\}$, where the map $\mu_A : \mathcal{U} \to [0,1]$ is called the* membership function *of A.*

Thus, the fuzzy set A is characterized by function μ_A. For every $x \in \mathcal{U}$, $\mu_A(x) \in [0,1]$ is a real number that describes the *degree of membership of x in A*. Also, if we observe that a ordinary set $A \subset \mathcal{U}$ is determined by the indicator function or *characteristic function* χ_A,

$$\chi_A : \mathcal{U} \to \{0,1\}, \quad \chi_A(x) = \begin{cases} 1, \text{ if } x \in A \\ 0, \text{ if } x \notin A \end{cases}$$

and, since the function μ_A is a generalization of the function χ_A, a fuzzy set is a generalization of the concept of an ordinary set or the notion of crisp set is extended by the corresponding notion of fuzzy set.

Given A, B fuzzy sets of an universe $\mathcal{U}$, A is said included in B (A is a subset of B) if, and only if, the membership function of A is less than that of B, that is, $A \subset B \Leftrightarrow \mu_A(x) \leq \mu_B(x), \forall x \in \mathcal{U}$.

If, in the above definition, we use a complete lattice L instead of interval $[0,1]$, then it arises the following concept of L-fuzzy set.

Definition 5. *[15] Let $(L, \leq)$ be a complete lattice. An L-*fuzzy set *A of an universe $\mathcal{U}$, is defined by the membership function $\mu_A : \mathcal{U} \to L$.*

In particular, we are interested in expressing the Herbrand base also as a L-fuzzy set, that is, if $\mathcal{B}_\mathcal{P} = \{A_1, \ldots, A_n, \ldots\}$ is the (crisp) Herbrand base of $\mathcal{P}$, we denote by $\mathcal{B}^L_\mathcal{P} = \{A_1|\top, \ldots, A_n|\top, \ldots\}$ the fuzzy Herbrand base and we have $\mu_{\mathcal{B}^L_\mathcal{P}} : \mathcal{F} \to L$ is such that $\mu_{\mathcal{B}^L_\mathcal{P}}(A) = \top = sup(L)$, if $A = A_i$, for any i, and $\mu_{\mathcal{B}^L_\mathcal{P}}(A) = \bot$, otherwise. Here, and thereafter, $\mathcal{F}$ denotes the set of all the formulae of the multi-adjoint language, namely, the set of all formulae generated by the set of symbols of a given multi-adjoint logic program $\mathcal{P}$.

In what follows we formulate, in an original way, the notion of fuzzy Herbrand model conceived as L-fuzzy set of the Herbrand base of the multi-adjoint program.

Definition 6. *A* fuzzy Herbrand interpretation[5] *$\mathcal{I}$ is a L-fuzzy set of the universe $\mathcal{B}_\mathcal{P}$ or, equivalently, a map $\mu_\mathcal{I} : \mathcal{B}_\mathcal{P} \to L$ (in fact, $\mu_\mathcal{I}$ is the membership function of L-fuzzy set), where $\mathcal{B}_\mathcal{P}$ is the Herbrand base of $\mathcal{P}$ and $(L, \leq)$ is the multi-adjoint lattice associated to $\mathcal{P}$.*

Indeed, the above function $\mu_\mathcal{I}$ can be extended in a natural way to the set of all formulas $\mathcal{F}$. In particular, for every (closed) formula $A \in \mathcal{F}$, $\mu_\mathcal{I}(A) = inf_\xi\{\mu_\mathcal{I}(A\xi) : A\xi$ is a ground instance of A$\}$.

[4] We follow the notation due to [21] expressing this pair by $x|\mu_A(x)$. It is customary to confuse the predicate $A(x)$ with the degree of membership $\mu_A(x)$, we prefer to explicitly distinguish these two concepts.

[5] We will also say Herbrand interpretation.

Definition 7. *A fuzzy Herbrand interpretation $\mathcal{I}$ satisfies (or is* Herbrand model *of) a rule $\mathcal{R}_i : \langle R_i; \alpha_i \rangle$ if, and only if, $\alpha_i \leq \mu_{\mathcal{I}}(\mathcal{R}_i)$. An Herbrand interpretation $\mathcal{I}$ is Herbrand model of $\mathcal{P}$ iff all rules in $\mathcal{P}$ are satisfied by $\mathcal{I}$.*

Obviously, if $\mathcal{I}$ is a Herbrand model of $\mathcal{P}$, we have $\mu_{\mathcal{I}}(A) \leq \top$, for all $A \in \mathcal{B}^L_{\mathcal{P}}$. Then, using Definition 6, $\mathcal{I}$ is a fuzzy subset of the Herbrand base $\mathcal{B}^L_{\mathcal{P}}$.

Let $\mathcal{H}^L$ be the set of Herbrand interpretations whose order, induced from the order of L, is given by $\mathcal{I}_j \subset \mathcal{I}_k \iff \mu_{\mathcal{I}_j}(F) \leq \mu_{\mathcal{I}_k}(F)$, for all formula $F \in \mathcal{F}$. It is easy to check that $(\mathcal{H}^L, \subset)$ inherits the structure of complete lattice from the multi-adjoint lattice $(L, \leq)$. Also, note that $\mathcal{H}^L$ is a set of L-fuzzy sets of universe $\mathcal{F}$.

It is important to observe that, using L-fuzzy sets, the least fuzzy Herbrand model for multi-adjoint logic programing, can be characterized by the following definition, that is, exactly the same terms as expressed in [6] for pure logic programming.

Definition 8. *Let $\mathcal{P}$ be a multi-adjoint logic program with associated lattice $(L, \leq)$. The L-fuzzy set $\mathcal{I}^L_{\mathcal{P}} = \bigcap \mathcal{I}_j$, where $\mathcal{I}_j$ is a Herbrand model of $\mathcal{P}$, is called* least fuzzy Herbrand model *of $\mathcal{P}$.*

The previous interpretation $\mathcal{I}^L_{\mathcal{P}}$ can be thought indeed as the least fuzzy Herbrand model, by virtue of the following result.

Theorem 1. *Let $\mathcal{P}$ be a multi-adjoint program with associated lattice $(L, \leq)$. Then, $\mathcal{I}^L_{\mathcal{P}} = \bigcap \mathcal{I}_j$, where $\mathcal{I}_j$ is a Herbrand model of $\mathcal{P}$, is the least Herbrand model of $\mathcal{P}$.*

Proof. Let $\mathcal{K}$ be the set of Herbrand model of $\mathcal{P}$, that is, the set $\mathcal{K} = \{\mathcal{I}_j : \mathcal{I}_j$ is a Herbrand model of $\mathcal{P}\}$. $\mathcal{I}^L_{\mathcal{P}}$ is a Herbrand interpretation by construction. Since $(\mathcal{H}^L, \subset)$ is a complete lattice, there exist the infimum of the set $\mathcal{K}$, it is a member of $\mathcal{H}^L$ and is given by the intersection of all Herbrand models I_j.

Moreover, $\mathcal{I}^L_{\mathcal{P}}$ is also a Herbrand model of $\mathcal{P}$. By definition of intersection, $\mathcal{I}^L_{\mathcal{P}} \subset \mathcal{I}_j$ for each Herbrand model $\mathcal{I}_j$ of $\mathcal{P}$. Therefore, $\mu_{\mathcal{I}^L_{\mathcal{P}}}(A) \leq \mu_{\mathcal{I}_j}(A)$ for each atom A. On the other hand, since each $\mathcal{I}_j$ is a model of $\mathcal{P}$, by definition of Herbrand model, each rule $\mathcal{R} : \langle A \leftarrow_i \mathcal{B}; v \rangle$ in $\mathcal{P}$ is satisfied by $\mathcal{I}_j$, that is, $v \leq \mu_{\mathcal{I}_j}(A \leftarrow_i \mathcal{B})$. Now, by definition of Herbrand interpretation, the monotonic properties of adjoint pairs in a multi-adjoint lattice, and because $\mu_{\mathcal{I}^L_{\mathcal{P}}}(A) \leq \mu_{\mathcal{I}_j}(A)$:

$$v \leq \mu_{\mathcal{I}_j}(A \leftarrow_i \mathcal{B}) = \mu_{\mathcal{I}_j}(A) \dot{\leftarrow}_i \mu_{\mathcal{I}_j}(\mathcal{B}) \leq \mu_{\mathcal{I}^L_{\mathcal{P}}}(A) \dot{\leftarrow}_i \mu_{\mathcal{I}_j}(\mathcal{B}),$$

where $\dot{\leftarrow}_i$ denote the truth function of the connective $\leftarrow_i$. By the adjoint property, $v \leq \mu_{\mathcal{I}^L_{\mathcal{P}}}(A) \dot{\leftarrow}_i \ \mu_{\mathcal{I}_j}(\mathcal{B})$ iff $v \dot{\&}_i \ \mu_{\mathcal{I}_j(}\mathcal{B}) \leq \mu_{\mathcal{I}^L_{\mathcal{P}}}(A)$. Also, since the operation $\dot{\&}_i$ is increasing in both arguments and $\mu_{\mathcal{I}^L_{\mathcal{P}}}(\mathcal{B}) \leq \mu_{\mathcal{I}_j}(\mathcal{B})$, $v \dot{\&}_i \ \mu_{\mathcal{I}^L_{\mathcal{P}}}(\mathcal{B}) \leq \mu_{\mathcal{I}^L_{\mathcal{P}}}(A)$. Also, applying the adjoint property once again, $v \dot{\&}_i \mu_{\mathcal{I}^L_{\mathcal{P}}(\mathcal{B})} \leq \mu_{\mathcal{I}^L_{\mathcal{P}}}(A)$ iff $v \leq \mu_{\mathcal{I}^L_{\mathcal{P}}}(A) \dot{\leftarrow}_i \ \mu_{\mathcal{I}^L_{\mathcal{P}}}(\mathcal{B}) = \mu_{\mathcal{I}^L_{\mathcal{P}}}(A \leftarrow_i \mathcal{B})$. Therefore, $\mathcal{I}^L_{\mathcal{P}}$ satisfies each rule $\mathcal{R}$ in $\mathcal{P}$, being a Herbrand model of $\mathcal{P}$.

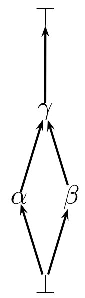
Trivially, since $\mathcal{I}^L_{\mathcal{P}}$ is the infimum of complete lattice $(\mathcal{K}, \subset)$ (indeed, $inf\{\mathcal{I}_j : \mathcal{I}_j$ is a Herbrand model$\}=\bigcap I_j$), it is the least Herbrand model of $\mathcal{P}$, which concludes the proof. □

Example 1. Consider the following multi-adjoint logic program $\mathcal{P}$ composed by facts (rules whose bodies are implicitly assumed to be $\top$) and an associate lattice $(L, \leq)$ described by the Hasse's diagram of the figure:

$\mathcal{R}_1 : \langle p(a) \leftarrow\ ;\ \alpha\rangle$

$\mathcal{R}_2 : \langle p(a) \leftarrow\ ;\ \beta\rangle$

$\mathcal{R}_3 : \langle q(a) \leftarrow\ ;\ \beta\rangle$



	$\mathcal{I}_1$	$\mathcal{I}_2$	$\mathcal{I}_3$	$\mathcal{I}_4$	$\mathcal{I}_5$	$\mathcal{I}_6$
$p(a)$	γ	γ	γ	$\top$	$\top$	$\top$
$q(a)$	β	γ	$\top$	β	γ	$\top$

Here, $(\&_{\texttt{G}}, \leftarrow_{\texttt{G}})$ is the pair of connectives following the *Gödel's intuitionistic* logic, whose truth functions are defined as:

$$\dot{\&}_{\texttt{G}}(x, y) = \inf\{x, y\} \quad \text{and} \quad \dot{\leftarrow}_{\texttt{G}}(y, x) = \begin{cases} \top, & \text{if } x \leq y \\ y, & \text{otherwise} \end{cases}$$

It is important to note that with these definitions, the pair $(\leftarrow_{\texttt{G}}, \&_{\texttt{G}})$ verifies the condition for conforming an adjoint pair regarding lattice $(L, \leq)$ of the figure above.

There exist six different Herbrand models (see $\mathcal{I}_1, \ldots, \mathcal{I}_6$ in the previous table) being $\mathcal{I}^L_{\mathcal{P}} = \mathcal{I}_1$ the least fuzzy Herbrand model. It is easy to see that $\mathcal{I}^L_{\mathcal{P}}$ is the L-fuzzy

$$\mathcal{I}^L_{\mathcal{P}} = \{p(a)|\gamma, q(a)|\beta\} \subset \mathcal{B}^L_{\mathcal{P}} = \{p(a)|\top, q(a)|\top\}$$

3 Correct Answers by Using Fuzzy Sets

In this section we study the characterization of the notion of correct answer based on L-fuzzy sets. Moreover, we see also for the new formulation of least Herbrand model, that this L-fuzzy set is (like in pure logic programming, see [6]) the set of formulas in the Herbrand base which follow logically from the formulas of the MALP program[6]. The following theorem shows this characterization for correct answer $\langle\lambda; \theta\rangle$.

[6] It is not difficult to prove that the fuzzy least Herbrand model coincides with the set of logical consequences, similarly to pure logic programming.

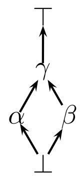
Theorem 2. *Let $\mathcal{P}$ be a multi-adjoint logic program and G a goal. The pair $\langle \lambda; \theta \rangle$ is a correct answer for $\mathcal{P}$ and G if, and only if, $\lambda \leq \mu_{\mathcal{I}_{\mathcal{P}}^L}(G\theta)$, where $\mathcal{I}_{\mathcal{P}}^L$ is the least fuzzy Herbrand model of $\mathcal{P}$, $\lambda \in L$ and θ is a substitution.*

Proof. Since $\mu_{\mathcal{I}_{\mathcal{P}}^L}$ is the membership function of $\mathcal{I}_{\mathcal{P}}^L$, it is enough to use the definitions of least fuzzy Herbrand model and correct answer. □

The following example adequately suggests how correct answers can be obtained from least fuzzy Herbrand model, the L-fuzzy set $\mathcal{I}_{\mathcal{P}}^L$.

Example 2. Consider the following program $\mathcal{P} = \{\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3\}$, whose associated lattice $(L, \leq)$ is given by its depicted Hasse diagram:

$$\mathcal{R}_1 : \langle p(a) \leftarrow \; ; \qquad \alpha \rangle$$
$$\mathcal{R}_2 : \langle p(b) \leftarrow \; ; \qquad \beta \rangle$$
$$\mathcal{R}_3 : \langle q(a) \leftarrow_{\mathtt{G}} p(a); \gamma \rangle$$



Following *Gödel*'s logic, the truth functions of connectives ($\leftarrow_{\mathtt{G}}, \&_{\mathtt{G}}$) are defined in Example 1, thus verifying the conditions for conforming an adjoint pair regarding lattice $(L, \leq)$ of the above figure.

The Herbrand base of the program $\mathcal{P}$ is $\mathcal{B}_{\mathcal{P}} = \{p(a), p(b), q(a), q(b)\}$, hence $\mathcal{B}_{\mathcal{P}}^L = \{p(a)|\top, p(b)|\top, q(a)|\top, q(b)|\top\}$. All Herbrand model of $\mathcal{P}$ is a fuzzy subset of $\mathcal{B}_{\mathcal{P}}^L$, in particular $\mathcal{I}_{\mathcal{P}}^L \subset \mathcal{B}_{\mathcal{P}}^L$. It is easy to check that the least Herbrand model $\mathcal{I}_{\mathcal{P}}^L$ can be given by the L-fuzzy set $\mathcal{I}_{\mathcal{P}}^L = \{p(a)|\alpha, p(b)|\beta, q(a)|\alpha,\ q(b)|\bot\}$. Then:

i) For goal $p(a)$ the set of correct answers is $\{\langle \lambda; id \rangle : \lambda \in L, \lambda \leq \alpha\}$.

ii) For goal $p(b)$ the set of correct answers is $\{\langle \lambda; id \rangle : \lambda \in L, \lambda \leq \beta\}$.

iii) For goal $q(a)$ the set of correct answers is $\{\langle \lambda; id \rangle : \lambda \in L, \lambda \leq \alpha\}$.

iv) For goal $p(x)$, the set of correct answers is $\{\langle \lambda; \theta \rangle : \lambda \in L, \lambda \leq \mu_{\mathcal{I}_{\mathcal{P}}^L}(p(x)\theta)\} = \{\langle \bot; \{x/a\}\rangle, \langle \alpha; \{x/a\}\rangle, \langle \bot; \{x/b\}\rangle, \langle \beta, \{x/b\}\rangle\}$.
Note that the membership of $p(x)$ to fuzzy set $\mathcal{I}_{\mathcal{P}}^L$ is
$\mu_{\mathcal{I}_{\mathcal{P}}^L}(p(x)) = \inf\{\mu_{\mathcal{I}_{\mathcal{P}}^L}(p(x)\sigma) : p(x)\sigma \text{ is ground}\} =^7 \inf\{\mu_{\mathcal{I}_{\mathcal{P}}^L}(p(a)), \mu_{\mathcal{I}_{\mathcal{P}}^L}(p(b))\} = \inf\{\alpha, \beta\} = \bot$.

v) For goal $q(x)$, the set of correct answers is $\{\langle \lambda; \theta \rangle : \lambda \in L, \lambda \leq \mu_{\mathcal{I}_{\mathcal{P}}^L}(q(x)\theta)\} = \{\langle \alpha; \{x/a\}\rangle, \langle \bot; \{x/b\}\rangle\}$.
We have now that $\mu_{\mathcal{I}_{\mathcal{P}}^L}(q(x)) = \inf\{\mu_{\mathcal{I}_{\mathcal{P}}^L}(q(a)), \mu_{\mathcal{I}_{\mathcal{P}}^L}(q(b))\} = \inf\{\alpha, \bot\} = \bot$ (it is easy justify that the least Herbrand model has to be defined this way from $q(a), q(b)$ formulae).

[7] Substitutions will only consider terms from the Herbrand universe of the program instead of variables.

In the following theorem we provide an original demonstration for the soundness of the procedural semantics of multi-adjoint programming. Therein we observe a certain analogy with the one included in [6] for the pure logic programming case, despite that the non refutational feature of our language and its fuzzy nature determine very significative differences between both ones. Before tackling the mentioned result, we state the following lemma, that has an instrumental character.

Lemma 1. *Let* $(L, \leq)$ *be a complete lattice. For all* A, B *subsets of* L, $A \subset B$, *implies* $inf(B) \leq inf(A)$.

Proof. It suffices to consider the definition of the infimum and the complete character of lattice $(L, \leq)$. □

Observe that, thanks to the previous lemma, we have $\mu_{\mathcal{I}}(A) \leq \mu_{\mathcal{I}}(A\theta)$[8], for all substitution θ and for all Herbrand interpretation $\mathcal{I}$, whenever the set of ground instances of formula $A\theta$ is a subset of the set of ground instances of A.

Theorem 3 (Soundness). *Let* $\mathcal{P}$ *be a multi-adjoint logic program,* A *an atomic goal and* $\langle\lambda; \theta\rangle$ *a fuzzy computed answer for* A *in* $\mathcal{P}$. *Then,* $\langle\lambda; \theta\rangle$ *is a correct answer for* $\mathcal{P}$ *and* A.

Proof. Let $D : [G_1, \ldots, G_n]$ be a derivation where $G_1 = \langle A; id\rangle \rightarrow^n_{AS/IS} \langle\lambda; \theta\rangle = G_n$. We prove the claim by induction on n, being n length of D.

We see that, in first place, the result holds for $n = 1$. Indeed, if for goal A exists the derivation $\langle A; id\rangle \rightarrow_{AS} \langle\lambda; \theta\rangle$, then rule $\mathcal{R} : \langle H \leftarrow_i; \lambda\rangle \in \mathcal{P}$ and $A\theta = H\theta$. In that case, every Herbrand model $\mathcal{I}$ of $\mathcal{P}$ satisfies rule $\mathcal{R}$ and, then, $\lambda \leq \mu_{\mathcal{I}}(H \leftarrow_i)$, namely, $\lambda \leq \mu_{\mathcal{I}}(H)$. Furthermore, from the equality $A\theta = H\theta$ it follows that $\mu_{\mathcal{I}}(A\theta) = \mu_{\mathcal{I}}(H\theta)$ and by Lemma 1, we obtain $\mu_{\mathcal{I}}(H) \leq \mu_{\mathcal{I}}(H\theta)$. Consequently, we have $\lambda \leq \mu_{\mathcal{I}}(A\theta)$ and $\langle\lambda; \theta\rangle$ is a correct answer for $\mathcal{P}$ and A, as wanted.

Next suppose that the result is true for all derivation with length k and let us see that it is verified for an arbitrary derivation of length $k + 1$, $D : [G_1, \ldots, G_{k+1}]$. Noting the first step of derivation D, we have $G_1 = \langle A; id\rangle \rightarrow_{AS} \langle v \&_i \mathcal{B}\sigma; \sigma\rangle = G_2$. That is, the admissible step has been executed using the program rule $\mathcal{R} : \langle H \leftarrow_i \mathcal{B}; v\rangle$, where atom A unifies with the head of rule $\mathcal{R}$ through the mgu σ. For each atom $B_i\sigma$[9] , $i = 1, \ldots, n$, of $\mathcal{B}\sigma$ exists a derivation whose length is less or equal to k, which gives the computed answer $\langle b_i; \tau_i\rangle$.

More precisely, taking into account that $Dom(\sigma) \cap Ran(B_i) = \emptyset$, D includes the following admisible/interpretive derivation steps[10]:

[8] See, for instance, paragraphs $iv), v)$ of Example 2.

[9] Without lost of generality, we can suppose that in the considered derivation all admissible steps are executed before applying interpretive steps.

[10] If $\mathcal{Q}$ is a goal and σ is a substitution, an *interpretive computation* is a state transition system, whose transition relation $\rightarrow_{IS} \subseteq (\mathcal{E} \times \mathcal{E})$ is defined as $\langle Q[@(r_1, r_2)]; \sigma\rangle \rightarrow_{IS} \langle Q[@(r_1,r_2)/\dot{@}(r_1,r_2)]; \sigma\rangle$ where $\dot{@}$ is the truth function of connective @. If $\mathcal{Q}$ is a goal not containing atoms, an *interpretive derivation* is a sequence $\langle Q; \sigma\rangle \rightarrow^*_{IS} \langle Q'; \sigma\rangle$ of arbitrary interpretive steps.

$$
\begin{array}{ll}
D :[\langle A; id\rangle & \rightarrow_{AS} \\
\langle v \&_i \mathcal{B}\sigma; \sigma\rangle & = \\
\langle v \&_i @(B_1\sigma, \ldots, B_n\sigma); \sigma\rangle & \rightarrow^{l_1}_{AS/IS} \\
\langle v \&_i @(b_1, \ldots,\ B_n\sigma); \sigma \circ \tau_1\rangle & \rightarrow^{l_n}_{AS/IS} \\
\langle v \dot{\&}_i \dot{@}(b_1, \ldots, b_n); \sigma \circ \tau_1 \circ \cdots \circ \tau_n\rangle & \rightarrow_{IS} \\
\langle v \dot{\&}_i b; \sigma \circ \tau\rangle & \rightarrow_{IS} \\
\langle \lambda; \theta\rangle]
\end{array}
$$

where $\tau = \tau_1 \circ \tau_2 \circ \cdots \circ \tau_n$, $\theta = \sigma \circ \tau$, $\lambda = v \dot{\&}_i b$, $l_1 + l_2 + \ldots + l_n = k - 2$ y $b = \dot{@}(b_1, \ldots, b_n)$, being @ the combination of all conjunctions, disjunctions and aggregators that links the elements $b_i \in L$ in order to obtain the correct answer $\langle b; \tau\rangle$ for program $\mathcal{P}$ and goal $\mathcal{B}\sigma$.

By the induction hypothesis, for each $B_i\sigma$, $\langle b_i; \tau_i\rangle$ is a correct answer and, then, $b_i \leq \mu_{\mathcal{I}}(B_i\sigma\tau_i)$, for all Herbrand interpretation $\mathcal{I}$ that is model of $\mathcal{P}$. In that case, from $b_i \leq \mu_{\mathcal{I}}(B_i\sigma\tau_i)$ it follows that $b \leq \mu_{\mathcal{I}}(\mathcal{B}\sigma)$ since $\mu_{\mathcal{I}}(\mathcal{B}\sigma)$ is obtained from $\mu_{\mathcal{I}}(B_i\tau_i)$ as a result of applying the truth functions of conjunctions, disjunctions or aggregators, being all them monotone in each component.

Then, the equality $A\sigma = H\sigma$ entails $A\theta = H\theta$ and, therefore, $\mu_{\mathcal{I}}(A\theta) = \mu_{\mathcal{I}}(H\theta)$. Besides, by firstly using Lemma 1 having into account later that $(\leftarrow_i, \&_i)$ is an adjoint pair, it results $\lambda = v \dot{\&}_i b \leq v \dot{\&}_i \mu_{\mathcal{I}}(\mathcal{B}\sigma) \leq \mu_{\mathcal{I}}(H) \leq \mu_{\mathcal{I}}(H\theta)$.

Consequently, $\lambda \leq \mu_{\mathcal{I}}(A\theta)$ and $\langle \lambda; \theta\rangle$ is a correct answer for program $\mathcal{P}$ and atom A, as claimed. □

4 Logical Consequences by Using Fuzzy Sets

Now, we present a concept strongly related with the developments seen in the core of the paper. We include our approach of fuzzy logical consequences via fuzzy sets in this appendix due to lack of space in the body of the work.

We formalize the concept of logical consequence in terms of the fuzzy set $\mathcal{I}^L_{\mathcal{P}}$ and we relate it with the notion of correct answer. Moreover, we prove that the least Herbrand model $\mathcal{I}^L_{\mathcal{P}}$ coincides with the set of formulae from the Herbrand base $\mathcal{B}^L_{\mathcal{P}}$ that are a logical consequence of the set of rules of a multi-adjoint program. This result allows to extend, for multi-adjoint framework, the classical and well-known formulation of least Herbrand model to logic programming.

In what follows, we propose to state a characterization of the concept of logical consequence through the least Herbrand model $\mathcal{I}^L_{\mathcal{P}}$. Moreover, from a conceptual standpoint, this characterization will be formulated in a completely similar to the classical case.

Given the multi-adjoint logic program $\mathcal{P} = \{\mathcal{R}_1, \dots, \mathcal{R}_n\}$ with $\mathcal{R}_i : \langle R_i; \alpha_i \rangle$, $i = 1, \dots, n$, we have that $\mathcal{A} = \langle A; \alpha \rangle$ is a logical consequence of $\mathcal{P}$ if, and only if, $\alpha_i \leq \mathcal{I}_j(R_i) \Rightarrow \alpha \leq \mathcal{I}_j(A), \forall i, j$. Now, the following theorem gives a characterization of this concept in terms of fuzzy set $\mathcal{I}^L_{\mathcal{P}}$.

Theorem 4. *Let $\mathcal{P}$ be a multi-adjoint logic program and $\mathcal{A} = \langle A; \alpha \rangle$ a multi-adjoint formula. $\mathcal{A}$ is a logical consequence of $\mathcal{P}$ if, and only if, $\mathcal{I}^L_{\mathcal{P}}$ is a Herbrand model of $\mathcal{A}$.*

Proof. It is enough to consider the definition of (least fuzzy Herbrand model) $\mathcal{I}^L_{\mathcal{P}}$ in order to obtain the equivalence: $\mathcal{A}$ is a logical consequence of $\mathcal{P}$ if, and only if, $\alpha \leq \mu_{\mathcal{I}^L_{\mathcal{P}}}(A)$. □

In the following results we relate the concepts of logical consequence and correct answer.

Theorem 5. *Let $\mathcal{P}$ be a multi-adjoint logic program and G a goal. If $\langle \lambda; \theta \rangle$ is a correct answer for $\mathcal{P}$ and G then $\langle G\theta; \lambda \rangle$ is a logical consequence of $\mathcal{P}$.*

Proof. Let $\mathcal{I}^L_{\mathcal{P}}$ be the least Herbrand model of $\mathcal{P}$ and see that $\mathcal{I}^L_{\mathcal{P}}$ is Herbrand model of $\langle G\theta; \lambda \rangle$. However, by definition of correct answer is verified that $\lambda \leq \mu_{\mathcal{I}^L_{\mathcal{P}}}(G\theta)$, as wanted. □

Theorem 6. *Let $\mathcal{P}$ be a multi-adjoint logic program and $\mathcal{A} = \langle A; \alpha \rangle$ a multi-adjoint formula such that A is a goal. If $\mathcal{A}$ is a logical consequence of $\mathcal{P}$, then the pair $\langle \alpha; id \rangle$ is a correct answer for $\mathcal{P}$ and A.*

Proof. By the Theorem 4, $\mathcal{I}^L_{\mathcal{P}}$ is Herbrand model of $\mathcal{A}$, so that $\alpha \leq \mu_{\mathcal{I}^L_{\mathcal{P}}}(A)$ and therefore $\langle \alpha; id \rangle$ is a correct answer for $\mathcal{P}$ and A as claimed. □

Theorem 7. *Let $\mathcal{P}$ be a multi-adjoint logic program and $\mathcal{A} = \langle A\theta; \alpha \rangle$ a multi--adjoint formula such that $A\theta$ is a goal. If $\mathcal{A}$ is a logical consequence of $\mathcal{P}$, then the pair $\langle \alpha; \theta \rangle$ is a correct answer for $\mathcal{P}$ and A.*

Proof. Analogous to the above theorem. □

The next result is a natural adaptation, to multi-adjoint logic programing, of the corresponding theorem of pure logic programming, (see [6]), which characterizes the least Herbrand model as the set of formulae from the Herbrand base that are logical consequences of the multi-adjoint program. In this theorem we express a formula multi-adjoint $\mathcal{A} = \langle A; \alpha \rangle$ as the pair $A|\alpha$ (α is degree of membership of A in fuzzy set $\{A|\alpha\}$). Observe that this syntax is also allowed for rules in multi-adjoint program.

Theorem 8. *Let $\mathcal{I}^L_{\mathcal{P}}$ be the least fuzzy Herbrand model of a multi-adjoint program $\mathcal{P}$ with associated lattice L. If we choose formulae $\mathcal{A} = A|\alpha$ with $\alpha = \mu_{\mathcal{I}_j}(A)$, for some Herbrand model $\mathcal{I}_j$, then $\mathcal{I}^L_{\mathcal{P}} = \{\mathcal{A} \in \mathcal{B}^L_{\mathcal{P}} : \mathcal{A}$ is a logical consequence of $\mathcal{P}\}$.*

Proof. If $\mathcal{A} \in \mathcal{I}^L_{\mathcal{P}} \subset \mathcal{B}^L_{\mathcal{P}}$, then $\alpha \leq \mu_{\mathcal{I}^L_{\mathcal{P}}}(A)$ so $\mathcal{A}$ is logical consequence of $\mathcal{P}$ and this shows that $\mathcal{I}^L_{\mathcal{P}} \subset \{\mathcal{A} \in \mathcal{B}^L_{\mathcal{P}} : \mathcal{A}$ is a logical consequence of $\mathcal{P}\}$. For the reverse inclusion, let $\mathcal{A} = A|\alpha$ be a formulae, with $\alpha = \mu_{\mathcal{I}_j}(A)$, for some Herbrand model $\mathcal{I}_j$; then, $\mu_{\mathcal{I}^L_{\mathcal{P}}}(A) \leq \mu_{\mathcal{I}_j}(A) = \alpha$, because $\mathcal{I}^L_{\mathcal{P}} \subset \mathcal{I}_j$. Moreover, since $\mathcal{A}$ is a logical consequence of $\mathcal{P}$, $\alpha \leq \mu_{\mathcal{I}^L_{\mathcal{P}}}(A)$ and, consequently, $\alpha = \mu_{\mathcal{I}^L_{\mathcal{P}}}(A)$, as required. □

5 Conclusions and Future Work

This paper has focused on the MALP framework, for which during the last years we have produced a wide range of results regarding both theoretical [2,3,4,13], and practical [11,12,1] developments. After recalling from [5] our concept of least Herbrand model for MALP, we have characterized, through the concept of fuzzy set, notions of Herbrand model, least Herbrand model and correct answer, thus extending the classic concepts of pure logic programming to this kind of fuzzy logic programs. The main goals of this work have been both the re-formulation of all these concepts as well as their strong relationships (by also including an original proof of the soundness for the procedural semantics of MALP) by means of the well-known fuzzy sets theory, thus providing more natural and clearer results which directly resemble the properties of pure logic programming described in [6], but lifted now to the modern case of fuzzy logic programming. We are nowadays implementing most notions defined in this paper inside our "*Fuzzy LOgic Programming Environment for Research*" $\mathcal{FLOPER}$ (visit `http://dectau.uclm.es/floper/` where some real-world examples are available too).

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References

1. Almendros-Jiménez, J.M., Luna Tedesqui, A., Moreno, G.: A Flexible XPath-based Query Language Implemented with Fuzzy Logic Programming. In: Bassiliades, N., Governatori, G., Paschke, A. (eds.) RuleML 2011 - Europe. LNCS, vol. 6826, pp. 186–193. Springer, Heidelberg (2011)
2. Julián, P., Moreno, G., Penabad, J.: On Fuzzy Unfolding. A Multi-adjoint Approach. Fuzzy Sets and Systems 154, 16–33 (2005)
3. Julián, P., Moreno, G., Penabad, J.: Operational/Interpretive Unfolding of Multi-adjoint Logic Programs. Journal of Universal Computer Science 12(11), 1679–1699 (2006)
4. Julián, P., Moreno, G., Penabad, J.: An Improved Reductant Calculus using Fuzzy Partial Evaluation Techniques. Fuzzy Sets and Systems 160, 162–181 (2009)
5. Julián, P., Moreno, G., Penabad, J.: On the declarative semantics of multi-adjoint logic programs. In: Cabestany, J., Sandoval, F., Prieto, A., Corchado, J.M. (eds.) IWANN 2009, Part I. LNCS, vol. 5517, pp. 253–260. Springer, Heidelberg (2009)

6. Lloyd, J.W.: Foundations of Logic Programming. Springer, Berlin (1987)
7. Medina, J., Ojeda-Aciego, M.: Multi-adjoint logic programming. In: Proc. of the 10th International Conference on Information Processing and Managment of Uncertainty in Knowledge-Based Systems, IPMU 2004, Perugia, pp. 823–830 (2004)
8. Medina, J., Ojeda-Aciego, M., Vojtáš, P.: Multi-adjoint logic programming with continuous semantics. In: Eiter, T., Faber, W., Truszczyński, M. (eds.) LPNMR 2001. LNCS (LNAI), vol. 2173, pp. 351–364. Springer, Heidelberg (2001)
9. Medina, J., Ojeda-Aciego, M., Vojtáš, P.: A procedural semantics for multi-adjoint logic programming. In: Brazdil, P.B., Jorge, A.M. (eds.) EPIA 2001. LNCS (LNAI), vol. 2258, pp. 290–297. Springer, Heidelberg (2001)
10. Medina, J., Ojeda-Aciego, M., Vojtáš, P.: Similarity-based Unification: a multi-adjoint approach. Fuzzy Sets and Systems 146, 43–62 (2004)
11. Morcillo, P.J., Moreno, G.: Programming with fuzzy logic rules by using the FLOPER tool. In: Bassiliades, N., Governatori, G., Paschke, A. (eds.) RuleML 2008. LNCS, vol. 5321, pp. 119–126. Springer, Heidelberg (2008)
12. Morcillo, P.J., Moreno, G., Penabad, J., Vázquez, C.: Fuzzy Computed Answers Collecting Proof Information. In: Cabestany, J., Rojas, I., Joya, G. (eds.) IWANN 2011, Part II. LNCS, vol. 6692, pp. 445–452. Springer, Heidelberg (2011)
13. Morcillo, P.J., Moreno, G., Penabad, J., Vázquez, C.: Dedekind-MacNeille completion and cartesian product of multi-adjoint lattices. International Journal of Computer Mathematics, 1–11 (2012)
14. Mukaidono, M., Shen, Z., Ding, L.: Fundamentals of fuzzy prolog. International Journal Approximate Reasoning 3(2), 179–193 (1989)
15. Nguyen, H.T., Walker, E.A.: A First Course in Fuzzy Logic, 3rd edn. Chapman & Hall/CRC, Boca Ratón, Florida (2006)
16. Pedrycz, W., Gomide, F.: Introduction to fuzzy sets. MIT Press, Cambridge (1998)
17. Shapiro, E.Y.: Logic programs with uncertainties: A tool for implementing rule-based systems. In: Proc. of the 8th International Joint Conference on Artificial Intelligence, IJCAI 1983, Karlsruhe, pp. 529–532. University of Trier (1983)
18. van Emden, M.H.: Quantitative deduction and its fixpoint theory. Journal of Logic Programming 3(1), 37–53 (1986)
19. Vojtáš, P.: Fuzzy Logic Programming. Fuzzy Sets and Systems 124(1), 361–370 (2001)
20. Vojtáš, P., Paulík, L.: Soundness and completeness of non-classical extended SLD-resolution. In: Herre, H., Dyckhoff, R., Schroeder-Heister, P. (eds.) ELP 1996. LNCS, vol. 1050, pp. 289–301. Springer, Heidelberg (1996)
21. Zadeh, L.A.: Fuzzy Sets. Information and Control 8(3), 338–353 (1965)